

MEASURE OF RELATIVE (P, Q) -TH ORDER BASED ON A GROWTH OF COMPOSITE ENTIRE FUNCTIONS

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ABSTRACT. We deduce some growth properties of composite entire functions in the light of their relative (p, q) th order by extending some results of J. Tu, Z. X. Chen and X. M. Zheng [13].

1. Background, fundamental definitions and notations

Let f be an entire function defined on a set of all complex numbers \mathbb{C} . The maximum modulus function M_f or $M_f(r)$ of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined as $M_f = \max_{|z|=r} |f(z)|$. If f is non-constant entire, then its maximum modulus function $M_f(r)$ is strictly increasing and continuous, and therefore there exists its inverse function $M_f^{-1} : (|f(0)|, \infty) \rightarrow (0, \infty)$ with $\lim_{s \rightarrow \infty} M_f^{-1}(s) = \infty$. Moreover, for given any two entire functions f and g the ratio $\frac{M_f(r)}{M_g(r)}$, as $r \rightarrow \infty$, is called *the growth of f with respect to g* in terms of their maximum moduli. Our notations are standard within the theory of Nevanlinna's value distribution of entire functions, and therefore we do not explain those in detail as available in [14]. In the sequel the following two notations are used:

$$\begin{aligned} \log^{[k]} x &= \log \left(\log^{[k-1]} x \right) \text{ for } k = 1, 2, 3, \dots, \\ \log^{[0]} x &= x, \end{aligned}$$

and

$$\begin{aligned} \exp^{[k]} x &= \exp \left(\exp^{[k-1]} x \right) \text{ for } k = 1, 2, 3, \dots, \\ \exp^{[0]} x &= x. \end{aligned}$$

Let us recall that Juneja, Kapoor and Bajpai [8] defined the (p, q) -th order and (p, q) -th lower order, respectively, of an entire function f as follows:

$$\rho_f(p, q) = \limsup_{r \rightarrow \infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}, \quad \text{and} \quad \lambda_f(p, q) = \liminf_{r \rightarrow \infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

where p, q are positive integers with $p \geq q$.

In this connection we just recall the following definition:

Definition 1.1. [8] *An entire function f is said to have index-pair (p, q) , $p \geq q \geq 1$ if $b < \rho_f(p, q) < \infty$ and $\rho_f(p-1, q-1)$ is not a nonzero finite number, where $b = 1$ if $p = q$, and $b = 0$ if $p > q$. Moreover, if $0 < \rho_f(p, q) < \infty$, then*

$$\begin{cases} \rho_f(p-n, q) = \infty & \text{for } n < p, \\ \rho_f(p, q-n) = 0 & \text{for } n < q, \\ \rho_f(p+n, q+n) = 1 & \text{for } n = 1, 2, \dots \end{cases}$$

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Similarly for $0 < \lambda_f(p, q) < \infty$, one can easily verify that

$$\begin{cases} \lambda_f(p - n, q) = \infty & \text{for } n < p, \\ \lambda_f(p, q - n) = 0 & \text{for } n < q, \\ \lambda_f(p + n, q + n) = 1 & \text{for } n = 1, 2, \dots \end{cases}$$

The definition of (p, q) -th order ((p, q) -th lower order, respectively), as initiated by Juneja, Kapoor and Bajpai [8], extends the notion of *generalized order* $\rho_f^{[l]}$ (*generalized lower order* $\lambda_f^{[l]}$, resp.) of an entire function f introduced by Sato [11] for each integer $l \geq 2$, as these correspond to the particular case $\rho_f^{[l]} = \rho_f(l, 1)$ ($\lambda_f^{[l]} = \lambda_f(l, 1)$, resp.). If $p = 2$ and $q = 1$, then we write $\rho_f(2, 1) = \rho_f$ ($\lambda_f(2, 1) = \lambda_f$, resp.) which is known as *order* (*lower order*, resp.) of an entire function f . The *order* (*lower order*, resp.) of an entire function f is classical in complex analysis and is generally used in computational purpose which is defined in terms of the growth of f with respect to the function $\exp z$ function as:

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log \log M_{\exp z}(r)} = \limsup_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log r}$$

$$\left(\lambda_f = \liminf_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log \log M_{\exp z}(r)} = \liminf_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log r}, \text{ resp.} \right).$$

Bernal [1, 2] introduced the relative order between two entire functions to avoid comparing growth just with $\exp z$ which is as follows:

Definition 1.2. [1, 2] *The relative order of f with respect to g , denoted as $\rho_g(f)$, is defined by:*

$$\begin{aligned} \rho_g(f) &= \inf \{ \mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0 \} \\ &= \limsup_{r \rightarrow \infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \end{aligned}$$

This definition coincides with the classical one if $g = \exp z$ [12]. Similarly, one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as

$$\lambda_g(f) = \liminf_{r \rightarrow \infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Lahiri and Banerjee [9] gave a more generalized concept of relative order in the following way:

Definition 1.3. [9] *If $k \geq 1$ is a positive integer, then the k -th generalized relative order of f with respect to g , denoted by $\rho_f^k(g)$ is defined by*

$$\begin{aligned} \rho_g^k(f) &= \inf \left\{ \mu > 0 : M_f(r) < M_g \left(\exp^{[k-1]} r^\mu \right) \text{ for all } r > r_0(\mu) > 0 \right\} \\ &= \limsup_{r \rightarrow \infty} \frac{\log^{[k]} M_g^{-1} M_f(r)}{\log r}. \end{aligned}$$

Clearly, $\rho_g^1(f) = \rho_g(f)$ and $\rho_{\exp}^1(f) = \rho_f$.

The following definition of relative (p, q) th order of an entire function in the light of index-pair is due to Sanchez Ruiz et. al. [10]:

Definition 1.4. [10] *Let f and g be any two entire functions with index-pairs (m, q) (and (m, p) resp.) where p, q, m are positive integers such that $m \geq \max(p, q)$. Then the relative (p, q) -th order of f with respect to g is defined as*

$$\rho_g^{(p, q)}(f) = \limsup_{r \rightarrow \infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}.$$

The relative (p, q) -th lower order of f with respect to g is defined by:

$$\lambda_g^{(p,q)}(f) = \liminf_{r \rightarrow \infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}.$$

The previous definitions are easily generated from above as particular cases, e.g. if f and g have got index-pair $(m, 1)$ and (m, k) , resp., then Definition 1.4 reduces to Definition 1.3. If the entire functions f and g have the same index-pair $(p, 1)$, where p is any positive integer, we get the definition of relative order introduced by Bernal [1], and if $g = \exp^{[m-1]} z$, then $\rho_g(f) = \rho_f^{[m]}$ and $\rho_g^{(p,q)}(f) = \rho_f^{(p,q)}(m, q)$. And, if f is an entire function with index-pair $(2, 1)$ and $g = \exp z$, then Definition 1.4 becomes the classical one given in [12].

In order to calculate the growth rates of entire functions, the notions of use of the growth indicators such as *order* and *lower order* are classical in complex analysis and during the past decades, several researchers have already been continuing their studies in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of *relative orders* and *relative lower orders* of entire functions as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of composite entire functions in the light of their relative orders and relative lower orders are the prime concern of this paper. In fact, some light has already been thrown on such type of works by Datta et. al. in [4, 5, 6] and [7]. Taking into account all these above, we discuss in this paper some growth properties of composite entire functions in the light of their *relative (p, q) th order* and *relative (p, q) th lower order*, after improving some results of J. Tu, Z. X. Chen and X. M. Zheng [13].

2. Some examples

In this section we present some examples of entire functions in connection with definitions given in the previous section.

Example 2.1 (Order of \exp). *Given any natural number m , the exponential function $f(z) = \exp z^m$ has got $M_f(r) = \exp r^m$. Therefore $\frac{\log^{[2]} M_f(r)}{\log r}$ is constantly equal to m and consequently,*

$$\rho_f = \lambda_f = m.$$

Example 2.2 (Generalized order). *Given any natural numbers l, m , the function $f(z) = \exp^{[l]} z^m$ has got $M_f(r) = \exp^{[l]} r^m$. Therefore $\frac{\log^{[k]} M_f(r)}{\log r}$ is constant for each natural $k \geq 2$, thereby following that*

$$\rho_f^{[l+1]} = \lambda_f^{[l+1]} = m,$$

but $\rho_f^{[k]} = \lambda_f^{[k]} = +\infty$ for $2 \leq k \leq l$, and $\rho_f^{[k]} = \lambda_f^{[k]} = 0$ for $k > l + 1$.

Example 2.3 (Index-pair). *Given any four positive integers k, n, p, q with $p \geq q$, the function $f(z) = \exp^{[k]} z^n$ generates a constant quotient $\frac{\log^{[p]} M_f(r)}{\log^{[q]} r}$, and clearly*

$$\rho_f(p, q) = \lambda_f(p, q) = n \text{ for } (p, q) = (k + 1, 1),$$

but

$$\rho_f(p, q) = \lambda_f(p, q) = \begin{cases} 1 & \text{for } (p, q) = (k + h, h) = 1, h \in \mathbb{N}, \\ \infty & \text{for } p \leq q + 1, \\ 0 & \text{for } p \geq q + 1. \end{cases}$$

Thus f is a regular function with growth $(k + 1, 1)$.

Example 2.4 (Relative (p, q) -th order between functions). Suppose $f(z) = \exp^k \{z^n\}$ and $g(z) = \exp^{[k]} \{z^m\}$ with k, m, n any three positive integers. Then f and g are regular functions with $(k+1, 1)$ -growth with

$$\rho_f(k+1, 1) = n, \quad \rho_g(k+1, 1) = m.$$

In order to find out their $(1, 1)$ relative order we evaluate that

$$\frac{\log M_g^{-1} M_f(r)}{\log r} = \frac{\log \frac{1}{m} \left\{ \log^{[k]} (\exp^{[k]} r^n) \right\}^{\frac{1}{m}}}{\log r}$$

which happens to be constant. By taking limits, we easily get

$$\rho_g^{(1,1)}(f) = \lambda_g^{(1,1)}(f) = \frac{n}{m}.$$

3. Growth of composite entire functions

First of all, we recall one related known property which will be needed in order to prove our results, as we see in the following lemma.

Lemma 3.1. [3] *If f and g are two entire functions, then for all sufficiently large values of r*

$$M_f \left(\frac{1}{8} M_g \left(\frac{r}{2} \right) - |g(0)| \right) \leq M_{f \circ g}(r) \leq M_f(M_g(r)).$$

Now we present the main results concerning the growth of the composite entire functions f and g .

Theorem 3.1. *Let f and g be any two entire functions with index-pairs (p, q) and (m, n) , resp., where p, q, m, n are all positive integers such that $p \geq q$ and $m \geq n$. Then*
(i) the index-pair of $f \circ g$ is (p, n) when $q = m$ and either $\lambda_f(p, q) > 0$ or $\lambda_g(m, n) > 0$. Also

$$\begin{aligned} (a) \quad & \lambda_f(p, q) \rho_g(m, n) \leq \rho_{f \circ g}(p, n) \leq \rho_f(p, q) \rho_g(m, n) \text{ if } \lambda_f(p, q) > 0, \text{ and} \\ (b) \quad & \lambda_f(p, q) \rho_g(m, n) \leq \rho_{f \circ g}(p, n) \leq \rho_f(p, q) \rho_g(m, n) \text{ if } \lambda_g(m, n) > 0; \end{aligned}$$

(ii) the index-pair of $f \circ g$ is $(p, q + n - m)$ when $q > m$, and either $\lambda_f(p, q) > 0$ or $\lambda_g(m, n) > 0$. Also

$$\begin{aligned} (a) \quad & \lambda_f(p, q) \leq \rho_{f \circ g}(p, q + n - m) \leq \rho_f(p, q) \text{ if } \lambda_f(p, q) > 0, \text{ and} \\ (b) \quad & \rho_{f \circ g}(p, q + n - m) = \rho_f(p, q) \text{ if } \lambda_g(m, n) > 0; \end{aligned}$$

(iii) the index-pair of $f \circ g$ is $(p + m - q, n)$ when $q < m$, and either $\lambda_f(p, q) > 0$ or $\lambda_g(m, n) > 0$. Also

$$\begin{aligned} (a) \quad & \rho_{f \circ g}(p + m - q, n) = \rho_g(m, n) \text{ if } \lambda_f(p, q) > 0, \text{ and} \\ (b) \quad & \lambda_g(m, n) \leq \rho_{f \circ g}(p + m - q, n) \leq \rho_g(m, n) \text{ if } \lambda_g(m, n) > 0. \end{aligned}$$

Proof. In view of the first part of Lemma 3.1, it follows for all sufficiently large values of r that

$$\log^{[p]} M_{f \circ g}(r) \geq (\lambda_f(p, q) - \varepsilon) \log^{[q]} M_g \left(\frac{r}{2} \right) + O(1), \quad (3.1)$$

and also for a sequence of values of r tending to infinity we have

$$\log^{[p]} M_{f \circ g}(r) \geq (\rho_f(p, q) - \varepsilon) \log^{[q]} M_g \left(\frac{r}{2} \right) + O(1). \quad (3.2)$$

Similarly, in view of the second part of Lemma 3.1, for all sufficiently large values of r we obtain

$$\log^{[p]} M_{f \circ g}(r) \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M_g(r). \quad (3.3)$$

Now, the following two cases may arise:

Case I. $q = m$.

From (3.3) for all sufficiently large values of r , we have

$$\begin{aligned} \log^{[p]} M_{f \circ g}(r) &\leq (\rho_f(p, q) + \varepsilon) (\rho_g(m, n) + \varepsilon) \\ \text{so that } \lim_{r \rightarrow \infty} \frac{\log^{[p]} M_{f \circ g}(r)}{\log^{[n]} r} &\leq \rho_f(p, q) \rho_g(m, n). \end{aligned} \quad (3.4)$$

Also from (3.1), for a sequence of values of r tending to infinity, we obtain

$$\begin{aligned} \log^{[p]} M_{f \circ g}(r) &\geq (\lambda_f(p, q) - \varepsilon) (\rho_g(m, n) - \varepsilon) \log^{[n]} r + O(1), \text{ hence} \\ \limsup_{r \rightarrow \infty} \frac{\log^{[p]} M_{f \circ g}(r)}{\log^{[n]} r} &\geq \lambda_f(p, q) \rho_g(m, n). \end{aligned} \quad (3.5)$$

Moreover, from (3.2) for a sequence of values of r tending to infinity, we get

$$\begin{aligned} \log^{[p]} M_{f \circ g}(r) &\geq (\rho_f(p, q) - \varepsilon) (\lambda_g(m, n) - \varepsilon) \log^{[n]} r + O(1), \text{ and} \\ \limsup_{r \rightarrow \infty} \frac{\log^{[p]} M_{f \circ g}(r)}{\log^{[n]} r} &\geq \rho_f(p, q) \lambda_g(m, n). \end{aligned} \quad (3.6)$$

Therefore for $\lambda_f(p, q) > 0$ and from (3.4) and (3.5), we see that

$$\begin{aligned} \lambda_f(p, q) \rho_g(m, n) &\leq \limsup_{r \rightarrow \infty} \frac{\log^{[p]} M_{f \circ g}(r)}{\log^{[n]} r} \leq \rho_f(p, q) \rho_g(m, n), \\ \text{i.e., } \lambda_f(p, q) \rho_g(m, n) &\leq \rho_{f \circ g}(p, n) \leq \rho_f(p, q) \rho_g(m, n). \end{aligned} \quad (3.7)$$

Likewise, (3.4) and (3.6) for $\lambda_g(m, n) > 0$ yields

$$\begin{aligned} \rho_f(p, q) \lambda_g(m, n) &\leq \limsup_{r \rightarrow \infty} \frac{\log^{[p]} M_{f \circ g}(r)}{\log^{[n]} r} \leq \rho_f(p, q) \rho_g(m, n) \\ \text{i.e., } \rho_f(p, q) \lambda_g(m, n) &\leq \rho_{f \circ g}(p, n) \leq \rho_f(p, q) \rho_g(m, n). \end{aligned} \quad (3.8)$$

Also from (3.7) and (3.8) one can easily verify that $\rho_{f \circ g}(p-1, n) = \infty$, $\rho_{f \circ g}(p, n-1) = 0$ and $\rho_{f \circ g}(p+1, n+1) = 1$, and therefore we obtain that the index-pair of $f \circ g$ is (p, n) when $q = m$, and either $\lambda_f(p, q) > 0$ or $\lambda_g(m, n) > 0$. Thus the first part of the theorem is established.

Case II. $q > m$.

Now, from (3.3) for all sufficiently large values of r , we obtain

$$\begin{aligned} \log^{[p]} M_{f \circ g}(r) &\leq (\rho_f(p, q) + \varepsilon) \log^{[q-m]} \log^{[m]} M_g(r) \\ \text{i.e., } \log^{[p]} M_{f \circ g}(r) &\leq (\rho_f(p, q) + \varepsilon) \log^{[q-m]} \left[(\rho_g(m, n) + \varepsilon) \log^{[n]} r \right] \\ \text{i.e., } \log^{[p]} M_{f \circ g}(r) &\leq (\rho_f(p, q) + \varepsilon) \log^{[q+n-m]} r + O(1), \end{aligned}$$

therefore

$$\lim_{r \rightarrow \infty} \frac{\log^{[p]} M_{f \circ g}(r)}{\log^{[q+n-m]} r} \leq \rho_f(p, q). \quad (3.9)$$

Also, from (3.1) for a sequence of values of r tending to infinity, we have

$$\begin{aligned} \log^{[p]} M_{f \circ g}(r) &\geq (\lambda_f(p, q) - \varepsilon) \log^{[q-m]} \left[(\rho_g(m, n) - \varepsilon) \log^{[n]} \left(\frac{r}{2} \right) \right] \\ &\quad + O(1) \end{aligned}$$

$$\text{i.e., } \log^{[p]} M_{f \circ g}(r) \geq (\lambda_f(p, q) - \varepsilon) \log^{[q-m+n]} r + O(1),$$

hence

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p]} M_{f \circ g}(r)}{\log^{[q+n-m]} r} \geq \lambda_f(p, q). \quad (3.10)$$

Further, for a sequence of values of r tending to infinity, (3.2) yields

$$\log^{[p]} M_{f \circ g}(r) \geq (\rho_f(p, q) - \varepsilon) \log^{[q-m]} \left[(\lambda_g(m, n) - \varepsilon) \log^{[n]} \left(\frac{r}{2} \right) \right] + O(1)$$

$$i.e., \log^{[p]} M_{f \circ g}(r) \geq (\rho_f(p, q) - \varepsilon) \log^{[q+n-m]} r + O(1),$$

so that

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p]} M_{f \circ g}(r)}{\log^{[q+n-m]} r} \geq \rho_f(p, q). \quad (3.11)$$

Therefore, from (3.9) and (3.10) for $\lambda_f(p, q) > 0$, we obtain

$$\begin{aligned} \lambda_f(p, q) &\leq \limsup_{r \rightarrow \infty} \frac{\log^{[p]} M_{f \circ g}(r)}{\log^{[q+n-m]} r} \leq \rho_f(p, q) \\ i.e., \lambda_f(p, q) &\leq \rho_{f \circ g}(p, q + n - m) \leq \rho_f(p, q). \end{aligned} \quad (3.12)$$

Likewise, for $\lambda_g(m, n) > 0$, (3.9) and (3.11) follows

$$\begin{aligned} \rho_f(p, q) &\leq \limsup_{r \rightarrow \infty} \frac{\log^{[p]} M_{f \circ g}(r)}{\log^{[q+n-m]} r} \leq \rho_f(p, q) \\ i.e., \rho_{f \circ g}(p, q + n - m) &= \rho_f(p, q). \end{aligned} \quad (3.13)$$

Hence, from (3.12) and (3.13), one can easily verify that $\rho_{f \circ g}(p - 1, q + n - m) = \infty$, $\rho_{f \circ g}(p, q + n - m - 1) = 0$, and $\rho_{f \circ g}(p + 1, q + n - m + 1) = 1$. Therefore we get that the index-pair of $f \circ g$ is $(p, q + n - m)$ when $q > m$ and either $\lambda_f(p, q) > 0$ or $\lambda_g(m, n) > 0$, and thus the second part of the theorem follows.

Case III. $q < m$.

For all sufficiently large values of r and by (3.3) we obtain

$$\begin{aligned} \log^{[p+m-q]} M_{f \circ g}(r) &\leq \log^{[m]} M_g(r) + O(1) \\ i.e., \log^{[p+m-q]} M_{f \circ g}(r) &\leq (\rho_g(m, n) + \varepsilon) \log^{[n]} r + O(1), \end{aligned}$$

so that

$$\lim_{r \rightarrow \infty} \frac{\log^{[p+m-q]} M_{f \circ g}(r)}{\log^{[n]} r} \leq \rho_g(m, n). \quad (3.14)$$

Also, from (3.1) for a sequence of values of r tending to infinity, we have

$$\begin{aligned} \log^{[p+m-q]} M_{f \circ g}(r) &\geq \log^{[m]} M_g \left(\frac{r}{2} \right) + O(1) \\ i.e., \log^{[p+m-q]} M_{f \circ g}(r) &\geq (\rho_g(m, n) - \varepsilon) \log^{[n]} r + O(1), \end{aligned}$$

therefore

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p+m-q]} M_{f \circ g}(r)}{\log^{[n]} r} \geq \rho_g(m, n). \quad (3.15)$$

Further, an application of (3.2) for a sequence of values of r tending to infinity gives

$$\begin{aligned} \log^{[p+m-q]} M_{f \circ g}(r) &\geq \log^{[m]} M_g \left(\frac{r}{2} \right) + O(1) \\ i.e., \log^{[p+m-q]} M_{f \circ g}(r) &\geq (\lambda_g(m, n) - \varepsilon) \log^{[n]} r + O(1), \end{aligned}$$

and so

$$\limsup_{r \rightarrow \infty} \frac{\log^{[p+m-q]} M_{f \circ g}(r)}{\log^{[n]} r} \geq \lambda_g(m, n). \quad (3.16)$$

Therefore, (3.14) and (3.15) applied for $\lambda_f(p, q) > 0$ implies

$$\begin{aligned} \rho_g(m, n) &\leq \frac{\log^{[p+m-q]} M_{f \circ g}(r)}{\log^{[n]} r} \leq \rho_g(m, n) \\ \text{i.e., } \rho_{f \circ g}(p+m-q, n) &= \rho_g(m, n). \end{aligned} \quad (3.17)$$

Similarly, (3.14) and (3.16) for $\lambda_g(m, n) > 0$ yields

$$\begin{aligned} \lambda_g(m, n) &\leq \limsup_{r \rightarrow \infty} \frac{\log^{[p+m-q]} M_{f \circ g}(r)}{\log^{[n]} r} \leq \rho_g(m, n) \\ \text{i.e., } \lambda_g(m, n) &\leq \rho_{f \circ g}(p+m-q, n) \leq \rho_g(m, n). \end{aligned} \quad (3.18)$$

An application of the relation (3.17) and (3.18) easily gives that $\rho_{f \circ g}(p+m-q-1, n) = \infty$, $\rho_{f \circ g}(p+m-q, n-1) = 0$ and $\rho_{f \circ g}(p+m-q+1, n+1) = 1$. Therefore we obtain that the index-pair of $f \circ g$ is $(p+m-q, n)$ when $q < m$ and either $\lambda_f(p, q) > 0$ or $\lambda_g(m, n) > 0$, and thus the third part of the theorem is established. \square

Remark 3.1. Theorem 3.1 can be treated as an extension of Theorem 3.1 and Theorem 3.2 of Tu, Chen and Zheng [13].

Theorem 3.2. Let f and g be any two entire functions with index-pairs (p, q) and (m, n) , resp., where p, q, m, n are all positive integers such that $p \geq q$ and $m \geq n$. Then

$$\begin{aligned} (i) \quad \lambda_f(p, q) \lambda_g(m, n) &\leq \lambda_{f \circ g}(p, n) \\ &\leq \min \{ \rho_f(p, q) \lambda_g(m, n), \lambda_f(p, q) \rho_g(m, n) \} \\ &\quad \text{if } q = m, \lambda_f(p, q) > 0 \text{ and } \lambda_g(m, n) > 0, \end{aligned}$$

$$(ii) \quad \lambda_{f \circ g}(p, q+n-m) = \lambda_f(p, q) \text{ if } q > m, \lambda_f(p, q) > 0 \text{ and } \lambda_g(m, n) > 0,$$

and

$$(iii) \quad \lambda_{f \circ g}(p+m-q, n) = \lambda_g(m, n) \text{ if } q < m, \lambda_f(p, q) > 0 \text{ and } \lambda_g(m, n) > 0.$$

Reasoning similarly as in the proof of the Theorem 3.1 one can easily deduce the conclusion of Theorem 3.2, and so its proof is omitted.

Theorem 3.3. Let f, g, h and k be any four entire functions with index-pairs (p, q) , (m, n) , (a, b) and (c, d) , resp., where a, b, c, d, p, q, m, n are all positive integers such that $a \geq b$, $c \geq d$, $p \geq q$ and $m \geq n$.

(i) If either $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p+m-q, q \geq n)$ holds and $\lambda_f(p, q) > 0$, $0 < \lambda_h^{(b, n)}(f \circ g) \leq \rho_h^{(b, n)}(f \circ g) < \infty$, $0 < \lambda_k^{(d, q)}(f) \leq \rho_k^{(d, q)}(f) < \infty$, then

$$\begin{aligned} \frac{\lambda_h^{(b, n)}(f \circ g)}{\rho_k^{(d, q)}(f)} &\leq \liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[q-n]} r)} \leq \frac{\lambda_h^{(b, n)}(f \circ g)}{\lambda_k^{(d, q)}(f)} \\ &\leq \limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[q-n]} r)} \leq \frac{\rho_h^{(b, n)}(f \circ g)}{\lambda_k^{(d, q)}(f)}, \end{aligned}$$

and

(ii) If $q > m$, $a = c = p$, $\lambda_f(p, q) > 0$, $0 < \lambda_h^{(b, q+n-m)}(f \circ g) \leq \rho_h^{(b, q+n-m)}(f \circ g) < \infty$ and $0 < \lambda_k^{(d, q)}(f) \leq \rho_k^{(d, q)}(f) < \infty$, then

$$\begin{aligned} \frac{\lambda_h^{(b, q+n-m)}(f \circ g)}{\rho_k^{(d, q)}(f)} &\leq \liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[m-n]} r)} \leq \frac{\lambda_h^{(b, q+n-m)}(f \circ g)}{\lambda_k^{(d, q)}(f)} \\ &\leq \limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[m-n]} r)} \leq \frac{\rho_h^{(b, q+n-m)}(f \circ g)}{\lambda_k^{(d, q)}(f)}. \end{aligned}$$

Proof. Assume, that either $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$ hold and $\lambda_f(p, q) > 0$. Then in view of Theorem 3.1, the index-pair of $f \circ g$ is (p, n) or $(p + m - q, n)$, resp., and therefore by Definition 1.4, $\rho_h^{(b,n)}(f \circ g)$ ($\lambda_h^{(b,n)}(f \circ g)$, resp.), and $\rho_k^{(d,q)}(f)$ ($\lambda_k^{(d,q)}(f)$, resp.) exist.

Now from the definition of $\rho_k^{(d,q)}(f)$ and $\lambda_h^{(b,n)}(f \circ g)$, for arbitrary positive ε , and for all sufficiently large values of r , we have

$$\log^{[b]} M_h^{-1} M_{f \circ g}(r) \geq \left(\lambda_h^{(b,n)}(f \circ g) - \varepsilon \right) \log^{[n]} r \quad (3.19)$$

and

$$\log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right) \leq \left(\rho_k^{(d,q)}(f) + \varepsilon \right) \log^{[n]} r. \quad (3.20)$$

Now from (3.19) and (3.20), it follows for all sufficiently large values of r , that

$$\frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right)} \geq \frac{\left(\lambda_h^{(b,n)}(f \circ g) - \varepsilon \right) \log^{[n]} r}{\left(\rho_k^{(d,q)}(f) + \varepsilon \right) \log^{[n]} r}.$$

Since ε ($\varepsilon > 0$) is arbitrary, we obtain that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right)} \geq \frac{\lambda_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)}. \quad (3.21)$$

For a sequence of values of r tending to infinity we have

$$\log^{[b]} M_h^{-1} M_{f \circ g}(r) \leq \left(\lambda_h^{(b,n)}(f \circ g) + \varepsilon \right) \log^{[n]} r, \quad (3.22)$$

and for all sufficiently large values of r

$$\log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right) \geq \left(\lambda_k^{(d,q)}(f) - \varepsilon \right) \log^{[n]} r. \quad (3.23)$$

Combining (3.22) and (3.23), for a sequence of values of r tending to infinity, we get

$$\frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right)} \leq \frac{\left(\lambda_h^{(b,n)}(f \circ g) + \varepsilon \right) \log^{[n]} r}{\left(\lambda_k^{(d,q)}(f) - \varepsilon \right) \log^{[n]} r}.$$

For arbitrary ε ($\varepsilon > 0$), it follows

$$\liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right)} \leq \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}. \quad (3.24)$$

Also, for a sequence of values of r tending to infinity, we obtain

$$\log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right) \leq \left(\lambda_k^{(d,q)}(f) + \varepsilon \right) \log^{[n]} r. \quad (3.25)$$

Applying (3.19) and (3.25), for a sequence of values of r tending to infinity, we get

$$\frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right)} \geq \frac{\left(\lambda_h^{(b,n)}(f \circ g) - \varepsilon \right) \log^{[n]} r}{\left(\lambda_k^{(d,q)}(f) + \varepsilon \right) \log^{[n]} r}.$$

As ε ($\varepsilon > 0$) is arbitrary, we get from above that

$$\limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right)} \geq \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}. \quad (3.26)$$

For all sufficiently large values of r we obtain

$$\log T_h^{-1} T_{f \circ g}(r) \leq \left(\rho_h^{(b,n)}(f \circ g) + \varepsilon \right) \log^{[n]} r. \quad (3.27)$$

Combining now (3.23) and (3.27), it follows for all sufficiently large values of r

$$\frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[q-n]} r)} \leq \frac{\left(\rho_h^{(b,n)}(f \circ g) + \varepsilon \right) \log^{[n]} r}{\left(\lambda_k^{(d,q)}(f) - \varepsilon \right) \log^{[n]} r},$$

and, therefore, for arbitrary $\varepsilon (> 0)$, we obtain

$$\limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[q-n]} r)} \leq \frac{\rho_h \left(f \rho_h^{(b,n)}(f \circ g) \circ g \right)}{\lambda_k^{(d,q)}(f)}. \quad (3.28)$$

Thus the first part of the theorem follows from (3.21), (3.24), (3.26), and (3.28).

Similarly, one can easily derive the second part of the theorem. \square

Reasoning along the same line as in the proof of the Theorem 3.3 we obtain:

Theorem 3.4. *Let f, g, h and l be any four entire functions with index-pairs (p, q) , (m, n) , (a, b) and (x, y) , resp., where a, b, p, q, m, n, x, y are all positive integers such that $a \geq b$, $p \geq q, m \geq n$ and $x \geq y$.*

(i) If either $(q = m = x, a = p)$ or $(q < m = x, a = p + m - q)$ holds and $\lambda_g(m, n) > 0$, $0 < \lambda_h^{(b,n)}(f \circ g) \leq \rho_h^{(b,n)}(f \circ g) < \infty$, $0 < \lambda_l^{(y,n)}(g) \leq \rho_l^{(y,n)}(g) < \infty$, then

$$\begin{aligned} \frac{\lambda_h^{(b,n)}(f \circ g)}{\rho_l^{(y,n)}(g)} &\leq \liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[y]} M_l^{-1} M_g(r)} \leq \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_l^{(y,n)}(g)} \\ &\leq \limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[y]} M_l^{-1} M_g(r)} \leq \frac{\rho_h^{(b,n)}(f \circ g)}{\lambda_l^{(y,n)}(g)}, \end{aligned}$$

and

(ii) If $q > m = x$, $a = p$, $\lambda_g(m, n) > 0$, $0 < \lambda_h^{(b,q+n-m)}(f \circ g) \leq \rho_h^{(b,q+n-m)}(f \circ g) < \infty$, $0 < \lambda_l^{(y,n)}(g) \leq \rho_l^{(y,n)}(g) < \infty$, then

$$\begin{aligned} \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\rho_l^{(y,n)}(g)} &\leq \liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(\exp^{[q-m]} r)}{\log^{[y]} M_l^{-1} M_g(r)} \leq \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\lambda_l^{(y,n)}(g)} \\ &\leq \limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(\exp^{[q-m]} r)}{\log^{[y]} M_l^{-1} M_g(r)} \leq \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\lambda_l^{(y,n)}(g)}. \end{aligned}$$

Theorem 3.5. *Let f, g, h and k be any four entire functions with index-pairs (p, q) , (m, n) , (a, b) and (c, d) , resp., where a, b, c, d, p, q, m, n are all positive integers with $a \geq b$, $c \geq d$, $p \geq q$ and $m \geq n$.*

(i) If either $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$ holds and $\lambda_f(p, q) > 0$, $0 < \rho_h^{(b,n)}(f \circ g) < \infty$, $0 < \rho_k^{(d,q)}(f) < \infty$, then

$$\liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[q-n]} r)} \leq \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)} \leq \limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[q-n]} r)},$$

and

(ii) If $q > m$, $a = c = p$, $\lambda_f(p, q) > 0$, $0 < \rho_h^{(b,q+n-m)}(f \circ g) < \infty$ and $0 < \rho_k^{(d,q)}(f) < \infty$, then

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[m-n]} r)} &\leq \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \\ &\leq \limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[m-n]} r)}. \end{aligned}$$

Proof. Let either $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$ hold, and also let $\lambda_f(p, q) > 0$. In view of Theorem 3.1, the index-pair of $f \circ g$ is (p, n) or $(p + m - q, n)$, resp. Hence by Definition 1.4, $\rho_h^{(b,n)}(f \circ g)$ and $\rho_k^{(d,q)}(f)$ exist, and from the definition of $\rho_k^{(d,q)}(f)$, for a sequence of values of r tending to infinity, we get

$$\begin{aligned} \log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right) &\geq \left(\rho_k^{(d,q)}(f) - \varepsilon \right) \log^{[n]} r \\ \text{i.e., } \log T_{P[h]}^{-1} T_{P[f]}(r) &\geq \left(\rho_k^{(d,q)}(f) - \varepsilon \right) \log^{[n]} r. \end{aligned} \quad (3.29)$$

Now from (3.27) and (3.29), for a sequence of values of r tending to infinity, it follows that

$$\frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right)} \leq \frac{\left(\rho_h^{(b,n)}(f \circ g) + \varepsilon \right) \log^{[n]} r}{\left(\rho_k^{(d,q)}(f) - \varepsilon \right) \log^{[n]} r}.$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right)} \leq \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)}. \quad (3.30)$$

Again, for a sequence of values of r tending to infinity, we obtain

$$\log^{[b]} M_h^{-1} M_{f \circ g}(r) \geq \left(\rho_h^{(b,n)}(f \circ g) - \varepsilon \right) \log^{[n]} r. \quad (3.31)$$

Combining (3.20) and (3.31), for a sequence of values of r tending to infinity, we get

$$\frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right)} \geq \frac{\left(\rho_h^{(b,n)}(f \circ g) - \varepsilon \right) \log^{[n]} r}{\left(\rho_k^{(d,q)}(f) + \varepsilon \right) \log^{[n]} r}.$$

For arbitrarily chosen $\varepsilon (> 0)$, it follows from the above

$$\limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f \left(\exp^{[q-n]} r \right)} \geq \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)}. \quad (3.32)$$

Thus the first part of the theorem follows from (3.30) and (3.32).

Analogously, the second part of the proof of the theorem can be derived. \square

The proof of the following theorem can be carried out as of the Theorem 3.5, therefore we omit the details.

Theorem 3.6. Let f, g, h and l be any four entire functions with index-pairs $(p, q), (m, n), (a, b)$ and (x, y) , resp., where a, b, p, q, m, n, x, y are all positive integers such that $a \geq b, p \geq q, m \geq n$ and $x \geq y$.

(i) If either $(q = m = x, a = p)$ or $(q < m = x, a = p + m - q)$ holds and $\lambda_g(m, n) > 0, 0 < \rho_h^{(b,n)}(f \circ g) < \infty, 0 < \rho_l^{(y,n)}(g) < \infty$, then

$$\liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[y]} M_l^{-1} M_g(r)} \leq \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_l^{(y,n)}(g)} \leq \limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[y]} M_l^{-1} M_g(r)},$$

and

(ii) If $q > m = x, a = p, \lambda_g(m, n) > 0, 0 < \rho_h^{(b,q+n-m)}(f \circ g) < \infty, 0 < \rho_l^{(y,n)}(g) < \infty$, then

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g} \left(\exp^{[q-m]} r \right)}{\log^{[y]} M_l^{-1} M_g(r)} &\leq \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_l^{(y,n)}(g)} \\ &\leq \limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g} \left(\exp^{[q-m]} r \right)}{\log^{[y]} M_l^{-1} M_g(r)}. \end{aligned}$$

The following theorem is a natural consequence of Theorem 3.3 and Theorem 3.5:

Theorem 3.7. *Let f, g, h and k be any four entire functions with index-pairs (p, q) , (m, n) , (a, b) (and (c, d) , resp.), where a, b, c, d, p, q, m, n are all positive integers such that $a \geq b$, $c \geq d$, $p \geq q$ and $m \geq n$.*

(i) If either $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$ holds and $\lambda_f(p, q) > 0$, $0 < \lambda_h^{(b,n)}(f \circ g) \leq \rho_h^{(b,n)}(f \circ g) < \infty$, $0 < \lambda_k^{(d,q)}(f) \leq \rho_k^{(d,q)}(f) < \infty$, then

$$\liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[q-n]} r)} \leq \min \left\{ \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \leq$$

$$\max \left\{ \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \leq \limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[q-n]} r)},$$

and

(ii) If $q > m$, $a = c = p$, $\lambda_f(p, q) > 0$, $0 < \lambda_h^{(b,q+n-m)}(f \circ g) \leq \rho_h^{(b,q+n-m)}(f \circ g) < \infty$ and $0 < \lambda_k^{(d,q)}(f) \leq \rho_k^{(d,q)}(f) < \infty$, then

$$\liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[m-n]} r)} \leq \min \left\{ \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \leq$$

$$\max \left\{ \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \leq \limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[d]} M_k^{-1} M_f(\exp^{[m-n]} r)}.$$

We omit the proof because of the similarity to the previous ones.

Analogously one may formulate the following theorem without its proof.

Theorem 3.8. *Let f, g, h and l be any four entire functions with index-pairs (p, q) , (m, n) , (a, b) (and (x, y) , resp.), where a, b, p, q, m, n, x, y are all positive integers such that $a \geq b$, $p \geq q$, $m \geq n$ and $x \geq y$.*

(i) If either $(q = m = x, a = p)$ or $(q < m = x, a = p + m - q)$ holds and $\lambda_g(m, n) > 0$, $0 < \lambda_h^{(b,n)}(f \circ g) \leq \rho_h^{(b,n)}(f \circ g) < \infty$, $0 < \lambda_l^{(y,n)}(g) \leq \rho_l^{(y,n)}(g) < \infty$, then

$$\liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[y]} M_l^{-1} M_g(r)} \leq \min \left\{ \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_l^{(y,n)}(g)}, \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_l^{(y,n)}(g)} \right\} \leq$$

$$\max \left\{ \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_l^{(y,n)}(g)}, \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_l^{(y,n)}(g)} \right\} \leq \limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r)}{\log^{[y]} M_l^{-1} M_g(r)},$$

and

(ii) If $q > m = x$, $a = p$, $\lambda_g(m, n) > 0$, $0 < \lambda_h^{(b,q+n-m)}(f \circ g) \leq \rho_h^{(b,q+n-m)}(f \circ g) < \infty$, $0 < \lambda_l^{(y,n)}(g) \leq \rho_l^{(y,n)}(g) < \infty$, then

$$\liminf_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(\exp^{[q-m]} r)}{\log^{[y]} M_l^{-1} M_g(r)} \leq \min \left\{ \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\lambda_l^{(y,n)}(g)}, \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_l^{(y,n)}(g)} \right\} \leq$$

$$\max \left\{ \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\lambda_l^{(y,n)}(g)}, \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_l^{(y,n)}(g)} \right\} \leq \limsup_{r \rightarrow \infty} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(\exp^{[q-m]} r)}{\log^{[y]} M_l^{-1} M_g(r)}.$$

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